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Math 564: Adv. Analysis 1 HOMEWORK 3 **Due: Oct 15 (Sun), 11:59pm**

1. Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_d$ be σ-algebras on sets $X_1, X_2, \dots, X_d$, and denote by $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_d$ the σ-algebra on $X_1 \times X_2 \times \cdots \times X_d$ generated by the sets of the form $A_1 \times A_2 \times \cdots \times A_d$.

(a) Prove that for all second countable topological spaces $X_1, X_2, \dots, X_d$,

$$\mathcal{B}(\textstyle\prod_{i=1}^d X_i) = \bigotimes_{i=1}^d \mathcal{B}(X_i).$$

(Here $\prod_{i=1}^d X_i$ is the product topology.) In particular, $\mathcal{B}(\mathbb{R}^d) = \bigotimes_{i=1}^d \mathcal{B}(\mathbb{R})$.

(b) Let $(X, \mathcal{A})$ and $(Y_i, \mathcal{B}_i)$, $i = 1, 2$, be measurable spaces, i.e., sets equipped with σ-algebras. Prove that for $(\mathcal{A}, \mathcal{B}_i)$-measurable functions $f_i : X \to Y_i$, the function $(f_1, f_2) : X \to Y_1 \times Y_2$ defined by $x \mapsto (f_1(x), f_2(x))$ is $(\mathcal{A}, \mathcal{B}_1 \otimes \mathcal{B}_2)$-measurable.

(c) Now let (X, μ) be a measure space[1] and conclude that if $f_1, f_2 : X \to \mathbb{R}$ are μ-measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \to \mathbb{R}$ is μ-measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ-measurable.

2. For an equivalence relation E on a set X, a **selector** is a function $s : X \to X$ that picks a point from each E-class, more precisely, $s(x) E x$ and $x E y \Leftrightarrow s(x) = s(y)$ for all $x, y \in X$. By a selector/transversal for a group action, we mean that for its orbit equivalence relation.

Let Γ be a countable group and (X, μ) be a second countable metric space equipped with a Borel measure μ. Let $\alpha : \Gamma \curvearrowright X$ be a Borel action of Γ, i.e. each group element $\gamma \in \Gamma$ acts as a Borel function from X to X. Prove:

(a) α admits a Borel transversal if and only if it admits a Borel selector.

HINT: For $\Leftarrow$, note that a selector is an idempotent, so $s(X) = \{x \in X : s(x) = x\}$ and note that the diagonal $\{(x, x) : x \in X\}$ is in $\mathcal{B}(X) \otimes \mathcal{B}(X)$ by Question 1(a).

(b) If μ is atomless (as in Sierpinski's theorem) and non-zero (i.e. $\mu(X) > 0$), and the action α is μ-null-preserving[2] and μ-ergodic, then α does not admit a μ-measurable transversal or a μ-measurable selector.

REMARK: This is a generalization of what we proved for $\mathbb{E}_{\mathbb{Q}}$ and $\mathbb{E}_0$.

(c) If α is μ-ergodic, then for every second countable Hausdorff topological space Y, every α-invariant[3] μ-measurable function $f : X \to Y$ is constant a.e., i.e., there a conull set $X' \subseteq X$ such that $f|_{X'}$ is constant.

HINT: Define an appropriate notion of "heaviness" for open subsets of Y in some countable open basis and intersect all heavy basic open sets: you will get down to a point (like in the proof of Kőnig's lemma).

[1] We suppress the σ-algebra from the notation if it is not important.

[2] This means that for each μ-measurable set $A \subseteq X$ and $\gamma \in \Gamma$, the set γA is μ-null if and only if A is μ-null.

[3] This means that the function is constant on each α-orbit.

3. Prove that universally measurable functions are closed under compositions. More precisely, if X, Y, Z are topological spaces, and $f : X \to Y$ and $g : Y \to Z$ are universally measurable functions, then $g \circ f : X \to Z$ is universally measurable.

4. Prove the main case of the Measure Isomorphism Theorem, namely: every atomless Borel probability measure μ on $[0,1]$ is isomorphic to the Lebesgue measure λ on $[0,1]$.

Hint: The function f_μ associated to μ is an isomorphism from $([0,1],\mu)$ to $([0,1],\lambda)$.

5. Prove that the atoms of σ-finite Borel measures on second countable Hausdorff topological spaces (e.g., separable metric spaces) are points, more precisely, each atom is of the form $\{x\} \cup Z$, where x is a point and Z is a null set.

Hint: Fix an atom and define an appropriate notion of smallness for basic open sets (in some fixed countable basis), so that removing the small ones we are left with a point.

Remark: This statement has nothing to do with topology, the general statement (proven similarly) is: In a σ-finite measure space $(X, \mathcal{B}, \mu)$, where $\mathcal{B}$ is countably generated and separates points[4], the atoms are points.

6. (Cantor sets[5]) In a topological space X, a **Cantor set** is a set $C \subseteq X$ homeomorphic to the Cantor space $2^{\mathbb{N}}$. (In particular, C is a compact subset of X of size continuum.)

(a) In a connected[6] metric space X (such as $\mathbb{R}^d$), prove every Cantor set has is closed and has empty interior; in particular, it is nowhere dense.

(b) The standard Cantor set in $[0,1]$ is the set $C := \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} C_s$, where each C_s is a closed interval defined inductively by setting $C_\emptyset := [0,1]$ and letting C_{s0} and C_{s1} be the bottom third and top third closed subintervals of the closed interval C_s (in particular, $\mathrm{lh}(C_{si}) = \frac{1}{3}\mathrm{lh}(C_s)$), for each $s \in 2^{<\mathbb{N}}$. In particular, $C_0 := [0, \frac{1}{3}]$ and $C_1 := [\frac{2}{3}, 1]$, $C_{00} := [0, \frac{1}{3^2}]$, $C_{01} := [\frac{2}{3^2}, \frac{1}{3}]$, $C_{10} := [\frac{2}{3}, \frac{7}{3^2}]$, and $C_{11} := [\frac{8}{3^2}, 1]$, etc. Prove that C is indeed a Cantor set and that C is Lebesgue null.

(c) Define a Cantor subset of $[0,1]$ of positive Lebesgue measure.

Hint: Note that in the standard Cantor set, $C_{s0} \sqcup C_{s1}$ occupies 2/3 of C_s regardless of $n := \mathrm{lh}(s)$. Change the construction so that for each n, the $(n+1)^{\text{th}}$ level occupies the p_n proportion of the n^{th} level and the sequence (p_n) goes to 1 fast enough.

7. Follow the steps below to build an example of a Borel (in fact, continuous) function $f : [0,1] \to [0,1]$ and a Lebesgue measurable function $g : [0,1] \to [0,1]$ such that the composition $g \circ f$ is not Lebesgue measurable.

(i) Prove that every Lebesgue measurable set A of positive measure contains a non-measurable subset.

Hint: Any transversal of $E_{\mathbb{Q}}|_A$, the proof is the same as for $A := [0,1]$ done in class.

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[4] We say that a collection $\mathcal{C}$ of subsets of X **separates points** if for each pair x, y of distinct points, there is a set $C \in \mathcal{C}$ containing exactly one of the points x, y.

[5] Most of this quesiont was meant for Homework 1, but I forgot.

[6] A topological space is **connected** if it has no clopen sets, other than the whole space and $\emptyset$.

(ii) Let C_0 and C_+ be Cantor sets in $[0,1]$ where C_0 is Lebesgue null, while C_+ has positive Lebesgue measure. Let $h : C_+ \to C_0$ be a homeomorphism and extend it to a Borel function $f : [0,1] \to [0,1]$ so that $f(C_+) = C_0$ and $f(C_+^c)$ are disjoint.

Remark: If C_0 and C_+ are constructed by removing middle open intervals at every step, like in Question 6(b), then f can be taken to be continuous, think how.

(iii) Let $Y \subseteq C_+$ be a Lebesgue non-measurable set and put $g := \mathbb{1}_{f(Y)}$. Observe that g is Lebesgue measurable, however $g \circ f$ is not.

8. Let (X, μ) be a measure space. Prove that the integral of simple functions is well-defined, i.e. for all μ-measurable sets $A_i, B_j \subseteq X$ and $a_i, b_j \in \mathbb{R}$,

$$\sum_{i<n} a_i \mathbb{1}_{A_i} = \sum_{j<m} b_j \mathbb{1}_{B_j} \text{ implies } \sum_{i<n} a_i \mu(A_i) = \sum_{j<m} b_j \mu(B_j).$$